

On the discrete bicycle transformation

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1 Introduction

The motivation for this paper comes from the study of a simple model of bicycle motion. The bicycle is modeled as an oriented segment in the plane of fixed length ℓ , the wheelbase of the bicycle. The motion is constrained so that the segment is always tangent to the path of the rear wheel; this non-holonomic constraint is due to the fact that the rear wheel is fixed on the frame, whereas the front wheel can steer. See [8, 10, 13, 15] and the references therein.

If the rear wheel path γ is prescribed, and the direction of motion is chosen, the front wheel path Γ is constructed by drawing the tangent segments of length ℓ to γ . Note that the rear track may have cusp: they occur when the steering angle equals 90° . Changing the direction of motion to the opposite yields another front track, say, Γ' . We say that the curves Γ and Γ' are in the *bicycle correspondence*.¹ See Figure 1.

If the front wheel path Γ is prescribed then the rear wheel follows a constant-distance pursuit curve, and its trajectory is uniquely determined, once the initial position of the bicycle is chosen. A monodromy map $M_{\Gamma,\ell}$ arises that assigns to every initial position of the bicycle its terminal position. If Γ is a closed curve then $M_{\Gamma,\ell}$ is a self-map of a circle of radius ℓ , uniquely defined up to conjugation. The bicycle monodromy $M_{\Gamma,\ell}$ is a Möbius transformation [9, 10, 13].

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¹One can also call this Darboux or Bäcklund transformation, but we shall use the “bicycle” terminology.

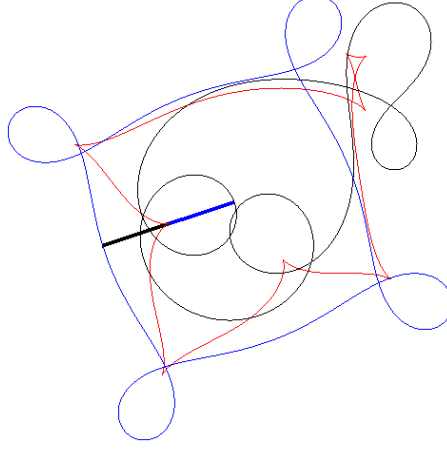


Figure 1: Bicycle correspondence. The cusped curve is the rear track, the two smooth curves are front tracks in the bicycle correspondence (figure courtesy of R. Perline).

All of the above can be extended to the motion of a segment in higher dimensional Euclidean spaces and even Riemannian manifolds (see [12] for elliptic and hyperbolic planes). In the forthcoming paper [16], we shall discuss Liouville integrability of the bicycle transformation in dimensions 2 and 3.

In this paper, following [11, 14], we study a discrete version of the bicycle correspondence. Let $V = (V_1, V_2, \dots)$ be a polygon in \mathbb{R}^n , and let V_1W_1 be a seed segment of length ℓ (so now ℓ is twice the length of the bicycle frame). The next point W_2 is constructed in the plane spanned by V_1, V_2, W_1 as follows: one parallel translates point W_1 along the vector V_1V_2 to point U , and then reflects point U in the line W_1V_2 to obtain a new point W_2 . In other words, the plane quadrilateral $V_1V_2W_1W_2$ is an isosceles trapezoid with $|V_1V_2| = |W_1W_2|$ and $|V_1W_1| = |V_2W_2| = \ell$, see Figure 2. Once the point W_2 is constructed, one continues the process, shifting the index by one, etc.²

We call the above described correspondence between polygons V and W the *discrete bicycle correspondence* and denote it by $\mathcal{B}_\ell(V, W)$. In the continuous limit, the polygons V and W become the front tire tracks Γ and Γ' , and the discrete bicycle correspondence becomes the above described

²The definition in [11, 14], given in 3-dimensional case, involves another, twist, parameter.

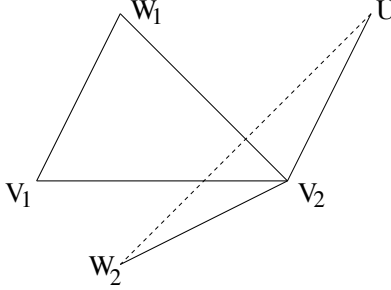


Figure 2: Discrete bicycle correspondence

bicycle correspondence between smooth curves.

Our ultimate goal is to establish Liouville integrability of the discrete bicycle correspondence and to describe its dynamics in detail. In this paper, we make steps in this direction. Let us list basic properties of the discrete bicycle correspondence.

Let V be a closed k -gon in \mathbb{R}^n (that is, $V_{i+k} = V_i$ for all i). The polygon W is not necessarily closed, and the discrete bicycle monodromy $M_{V,\ell}$ arises, similarly to the continuous case.

Theorem 1 *The monodromy $M_{V,\ell} : S^{n-1} \rightarrow S^{n-1}$ is a Möbius transformation of the sphere of radius ℓ .*

Thus, fixed points of the monodromy $M_{V,\ell}$ correspond to closed polygons W in the discrete bicycle correspondence with V .

Theorem 2 *Let V and W be closed polygons in \mathbb{R}^n in the discrete bicycle correspondence. Then, for every λ , the monodromies $M_{V,\lambda}$ and $M_{W,\lambda}$ are conjugated to each other.*

Theorem 2 implies that the invariants of the conjugacy class of the monodromy, viewed as functions of the “spectral parameter” λ , are integrals of the discrete bicycle correspondence. We refer to them as the monodromy integrals.

The next theorem states that the discrete bicycle correspondences with different length parameters commute with each other (“Bianchi permutability”). Recall that we write $\mathcal{B}_\ell(V, W)$ to indicate that polygons V and W are in the discrete bicycle correspondence with the length parameter ℓ .

Theorem 3 *Let V, W, S be closed k -gons in \mathbb{R}^n such that $\mathcal{B}_\ell(V, W)$ and $\mathcal{B}_\lambda(V, S)$ hold. Then there exists a closed polygon T such that $\mathcal{B}_\ell(S, T)$ and $\mathcal{B}_\lambda(W, T)$ hold.*

In the case of 3-dimensional space, Theorems 1-3 are not new: in [14], they are proved using quaternions. We give different proofs in Section 2.

V. Adler [1, 2] studied complete integrability of a correspondence on the space of polygons in Euclidean space called the *recutting of polygons*. The recutting R_i of polygon V at i th vertex is the reflection of V_i in the perpendicular bisector hyperplane of the segment $V_{i-1}V_{i+1}$. Recuttings of k -gons form a group with generators R_i , $i = 1, \dots, k$ and the relations

$$R_i^2 = 1, R_i R_j = R_j R_i \text{ for } |i - j| \geq 2, \text{ and } R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1},$$

where the indices are understood cyclically.

The recutting is closely related to the discrete bicycle correspondence. In Section 3, we show that certain integrals of the recutting, discovered by Adler, are integrals of the discrete bicycle correspondence. To do so, we construct a discrete analog of the rear track trajectory, a chain of mutually tangent spheres.

We also have the following result relating the discrete bicycle correspondence and the recutting.

Theorem 4 *1) The monodromy is preserved by the recutting. In particular, the monodromy integrals are also integrals of the recutting.
2) The discrete bicycle correspondence commutes with the recutting.*

To illustrate the first claim of Theorem 4, a parallelogram and the corresponding kite have the same monodromy, see Figure 3.

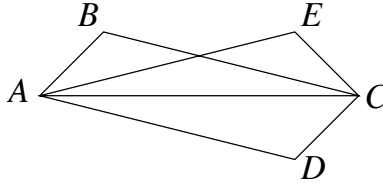


Figure 3: The parallelogram $ABCD$ and the kite $AECD$ have the same monodromy

Theorems 1-4 are proved in Section 2.

Consider the low-dimensional situation. If the dimension equals 2 then the discrete bicycle monodromy belongs to $SL(2, \mathbb{R})$. Then one has the trichotomy: $M_{V,\ell}$ may be elliptic, parabolic, and hyperbolic. In the last case, $M_{V,\ell}$ has two fixed points, and one can choose one (say, the attracting one) to construct a closed polygon W in the discrete bicycle correspondence with V (with length parameter ℓ). According to Theorem 2, $M_{W,\ell}$ is again hyperbolic, and one may iterate the construction by choosing the other fixed point of $M_{W,\ell}$ (otherwise, one gets back to V). Thus, the discrete bicycle correspondence becomes a map on polygons, and we write $\mathcal{T}_\ell(V) = W$.

In dimension three, the discrete bicycle monodromy belongs to $SL(2, \mathbb{C})$. If the monodromy is not the identity, it has two fixed points (perhaps, coinciding), and once again, one can consider the discrete bicycle correspondence as a mapping of the space of polygons in \mathbb{R}^3 .

In Sections 4 and 5, we study the discrete bicycle transformation on plane polygons. We prove that the discrete bicycle transformation is defined on convex cyclic polygons only if the length parameter does not exceed the diameter of the circumcircle, and in this case, the transformation is a rotation about the circumcenter. We also compute the eigenvalues of the discrete bicycle monodromy and derive a criterion for the monodromy to be parabolic.

In Section 5, we give a complete description of the dynamics of the discrete bicycle transformation on plane quadrilaterals. As an application, we classify the so-called bicycle $(4k, k)$ -gons (see Section 5 for definition).

2 Proofs of basic properties

Proof of Theorem 1. Recall that the Möbius group $O(n, 1)$ consists of linear isometries of the pseudo-Euclidean space $\mathbb{R}^{n,1}$, and it acts projectively on S^{n-1} , the spherization of the null cone; it is also the group of isometries of n -dimensional hyperbolic space (in the hyperboloid model).

Let M be the monodromy along segment V_1V_2 in Figure 2. We need to show that $M \in O(n, 1)$.

Let u, v and x be the unit vectors along V_1W_1, V_2W_2 and V_1V_2 , respectively, and let $|V_1V_2| = a$. The reflection of vector u in vector ξ is given by the formula

$$v = \frac{2u \cdot \xi}{|\xi|^2} \xi - u.$$

Applying this to $\xi = ax - \ell u$, we obtain

$$v = \frac{u + \frac{2a^2(x \cdot u)}{\ell^2 - a^2}x - \frac{2a\ell}{\ell^2 - a^2}x}{\frac{\ell^2 + a^2}{\ell^2 - a^2} - \frac{2a\ell(x \cdot u)}{\ell^2 - a^2}}. \quad (1)$$

On the other hand, a matrix from $O(n, 1)$ has the form

$$\begin{pmatrix} A & \xi \\ \eta^t & \lambda \end{pmatrix}$$

where A is an $n \times n$ matrix, ξ and η are n -vectors, and the following relations hold:

$$A^t A = E + \eta \otimes \eta^t, \quad A^t(\xi) = \lambda \eta, \quad \xi \cdot \xi = \lambda^2 - 1,$$

where E is the unit matrix, and $\eta \otimes \eta^t$ is the rank one matrix obtained by multiplying a column and a row vectors. The projective action of such a matrix is given by the formula:

$$u \mapsto \frac{A(u) + \xi}{\eta \cdot u + \lambda}. \quad (2)$$

We observe that (1) has the form (2) with

$$A = E + \frac{2a^2}{\ell^2 - a^2}x \otimes x, \quad \xi = \eta = -\frac{2a\ell}{\ell^2 - a^2}x, \quad \lambda = \frac{\ell^2 + a^2}{\ell^2 - a^2},$$

which completes the proof. \square

In dimension two, one identifies the unit circle with the real projective line via stereographic projection from point $(-1, 0)$. Then Möbius transformations become fractional-linear. If α is the angular coordinate on S^1 then $x = \tan(\alpha/2)$ is the respective affine coordinate on \mathbb{RP}^1 . In Figure 2, assume that $V_1 V_2$ is horizontal, the direction of $V_1 W_1$ is α and that of $V_2 W_2$ is β . If $x = \tan(\alpha/2)$ and $y = \tan(\beta/2)$ then the monodromy is given by the formula

$$y = \frac{\ell + a}{\ell - a}x,$$

or

$$M_\ell = \begin{pmatrix} \ell + a & 0 \\ 0 & \ell - a \end{pmatrix}.$$

In general, if the direction of V_1V_2 is ϕ then

$$M_\ell = \begin{pmatrix} \ell + a \cos \phi & -a \sin \phi \\ -a \sin \phi & \ell - a \cos \phi \end{pmatrix}. \quad (3)$$

Now we prove a property of isosceles trapezoids that is fundamental for what follows. Let $ABCD$ be a plane isosceles trapezoid, see Figure 4. We call the closed quadrilateral $ABDC$, made of the lateral sides and diagonals of a trapezoid, a *Darboux butterfly*.

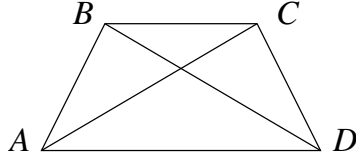


Figure 4: A Darboux butterfly

Lemma 2.1 (Butterfly Lemma) *The monodromy (with any length parameter ℓ) along a Darboux butterfly is the identity. Conversely, if the monodromy along a closed quadrilateral is the identity for some value of ℓ then the quadrilateral is a Darboux butterfly.*

Proof. The first statement of the lemma is 3-dimensional: if w is a test vector at vertex A then the respective vectors at all other vertices (the “transports” of w along the quadrilateral) belong to the 3-dimensional space, spanned by the plane of the trapezoid and the vector w .

In fact, it suffices to consider the case when w is in the plane of the trapezoid. Indeed, in dimension three, the monodromy is considered as an orientation preserving isometry of hyperbolic space acting on the sphere at infinity. If such an isometry has more than two fixed points then it is the identity.

In dimension two, we shall prove that the monodromy along the polygonal path ABD equals the monodromy along the path ACD if and only if $ABDC$ is a Darboux butterfly. Without loss of generality, assume that AD is horizontal. Let a, b, c, d be the length of the segments AB, BD, AC, CD , and let $\alpha, \beta, \gamma, \delta$ be the angles made with the positive horizontal axis. Let $|AD| = g$.

The product of the matrices from equation (3) is

$$\begin{pmatrix} \ell - b \cos \beta & -b \sin \beta \\ -b \sin \beta & \ell + b \cos \beta \end{pmatrix} \begin{pmatrix} \ell - a \cos \alpha & -a \sin \alpha \\ -a \sin \alpha & \ell + a \cos \alpha \end{pmatrix},$$

so we have the monodromy

$$M(a, b, \alpha, \beta) = \begin{pmatrix} \ell^2 - \ell g + ab \cos(\alpha - \beta) & -ab \sin(\alpha - \beta) \\ ab \sin(\alpha - \beta) & \ell^2 + \ell g + ab \cos(\alpha - \beta) \end{pmatrix}. \quad (4)$$

For equality to hold, we must have

$$M(a, b, \alpha, \beta) = k(\ell)M(c, d, \gamma, \delta)$$

for some constant $k(\ell)$ dependent only on ℓ . Therefore

$$\frac{\ell^2 - \ell g + ab \cos(\alpha - \beta)}{\ell^2 - \ell g + cd \cos(\gamma - \delta)} = \frac{ab \sin(\alpha - \beta)}{cd \sin(\gamma - \delta)} = \frac{\ell^2 + \ell g + ab \cos(\alpha - \beta)}{\ell^2 + \ell g + cd \cos(\gamma - \delta)}.$$

Set $X = \ell^2 + ab \cos(\alpha - \beta)$ and $Y = \ell^2 + cd \cos(\gamma - \delta)$. Then

$$\frac{X - \ell g}{Y - \ell g} = \frac{X + \ell g}{Y + \ell g},$$

hence $X = Y$ and

$$ab \cos(\alpha - \beta) = cd \cos(\gamma - \delta), \quad ab \sin(\alpha - \beta) = cd \sin(\gamma - \delta). \quad (5)$$

The second equation (5) implies that the signed area of triangle ABD is equal to that of triangle ACD , so that the quadrilateral $ABDC$ has a total signed area of zero. It also follows that $\tan(\alpha - \beta) = \tan(\gamma - \delta)$, so that $\alpha - \beta = \gamma - \delta$ or $\alpha - \beta = \gamma - \delta \pm \pi$. Since the signed areas are equal, the angles must be equal, and it follows that the quadrilateral is cyclic, and thus a Darboux butterfly.

Note that if the equality holds for one (non-zero) value of ℓ then it holds for all values of ℓ .

Finally, consider a non-planar quadrilateral $ABDC$ with the trivial monodromy (for some value of ℓ). Assume that the monodromy along ABD and ACD are equal. Denote this monodromy by M . Then M preserves the segments that lie in the plane ABD and in the plane ACD , and hence, in their intersection, the line AD . In the plane ABD , the monodromy M is given by

formula (4). If the horizontal axis is an eigendirection then $ab \sin(\alpha - \beta) = 0$. This implies that the segments AB and BD are collinear, a contradiction. \square

As a consequence of Butterfly Lemma, for every n , we can construct a family of $2n$ -gons with identity monodromy for all values of ℓ . These polygons are obtained by attaching Darboux butterflies to each other along the common sides, see Figure 5.

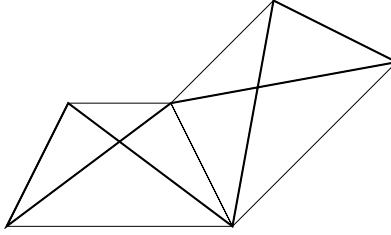


Figure 5: Constructing polygons with identity monodromy

Now we are in a position to prove the rest of the theorems.

Proof of Theorem 2. It follows from the Butterfly Lemma that, in Figure 2, one has:

$$M_{W_1W_2,\lambda} = M_{V_2W_2,\lambda} M_{V_1V_2,\lambda} M_{V_1W_1,\lambda}^{-1}.$$

Taking the composition over the closed polygon V yields the result. \square

Proof of Theorem 3. Consider the points V_1, W_1, S_1 , and let T_1 be the point such that $V_1W_1T_1S_1$ is a Darboux butterfly. Consider the discrete bicycle transformation of the segment V_1V_2 along the Darboux butterfly $V_1W_1T_1S_1$. According to the Butterfly Lemma, the resulting quadrilateral, say, Q , is closed and, according to Theorem 2, it has the trivial monodromy (for any length parameter). Hence, by the Butterfly Lemma again, Q is a Darboux butterfly as well.

It is clear from Figure 2 that the discrete bicycle transformation of the segment V_1W_1 along V_1V_2 is the same as the discrete bicycle transformation of the segment V_1V_2 along V_1W_1 . It follows that three of the vertices of Q are V_2, W_2 and S_2 . Denote the fourth vertex by T_2 .

A continuation of this process yields a closed polygon T satisfying the assertion of the theorem. \square

Proof of Theorem 4. An equivalent description of recutting $V_i \mapsto V'_i$ is that the quadrilateral $V_{i-1}V_iV_{i+1}V'_i$ is a Darboux butterfly.

To prove the first statement, we use Butterfly Lemma:

$$M_{V_{i-1}V'_iV_{i+1},\lambda} = M_{V_{i-1}V_iV_{i+1}V'_iV_{i+1},\lambda} = M_{V_{i-1}V_iV_{i+1},\lambda}.$$

For the second statement, let W be a polygon in the discrete bicycle correspondence with V . Let $V'_iW'_i$ be the discrete bicycle transformation of the segment $V_{i-1}W_{i-1}$ along the segment $V_{i-1}V'_i$. Since $V_{i-1}V_iV_{i+1}V'_i$ is a Darboux butterfly, the discrete bicycle transformation takes $V'_iW'_i$ to $V_{i+1}W_{i+1}$. Thus the polygon $\dots W_{i-1}W'_iW_{i+1}\dots$ is in the discrete bicycle correspondence with $\dots V_{i-1}V'_iV_{i+1}\dots$.

We want to show that the recutting of W on i th vertex yields W'_i or, equivalently, that $W_{i-1}W_iW_{i+1}W'_i$ is a Darboux butterfly. According to Butterfly Lemma, we need to show that the monodromy along the closed polygon $W_{i-1}W_iW_{i+1}W'_i$ is the identity. Indeed, using that the monodromy of each Darboux butterfly is trivial, we obtain:

$$\begin{aligned} M_{W_{i-1}W_iW_{i+1}W'_iW_{i-1},\lambda} &= M_{W_{i-1}V_{i-1}V_iW_iV_{i+1}W_{i+1}V_{i+1}V'_iW'_iV'_iV_{i-1}W_{i-1},\lambda} \\ &= M_{W_{i-1}V_{i-1}V_iV_{i+1}V'_iV_{i-1}W_{i-1},\lambda} = Id, \end{aligned}$$

and we are done. \square

3 Integrals

As we mentioned earlier, the discrete bicycle transformation preserves the conjugacy equivalence class of the monodromy M_λ , thus yielding the monodromy integrals. These integrals do not change if a polygon is acted upon by an isometry of the ambient space. We plan to study the monodromy integrals in a forthcoming paper. In this section, we study the integrals introduced in [1, 2] as integrals of the recutting. One of these integrals, $J(V)$, is not preserved by isometries. The other integral, $A(V)$, was described, in the 3-dimensional case, in [14].

Given a closed polygon V , consider the vector J and the bivector A given by the formulas

$$\begin{aligned} J(V) &= \sum_i (|V_{i+1}|^2 - |V_{i-1}|^2) V_i = \sum_i |V_i|^2 (V_{i-1} - V_{i+1}), \\ A(V) &= \sum_i V_i \wedge V_{i+1}, \end{aligned} \tag{6}$$

where the sums are cyclic. In dimension 2, $A(V)$ is the signed area of the polygon V .

Theorem 5 *Both A and J are integrals of the discrete bicycle transformation.*

As a preparation to the proof, we describe a discrete counterpart to the rear bicycle track (the middle curve with cusps in Figure 1).

We shall consider collections of spheres such that the first one is tangent to the second, the second to the third, \dots , and the last one is tangent to the first. We call such a collection a *chain*. The radii of the spheres are signed. By convention, if two spheres have an exterior tangency then their radii have the same sign, and if the tangency is interior then the radii have the opposite signs. We allow infinite radii, that is, we consider hyperplanes as spheres as well. An infinite radius has no sign (equivalently, one may consider the curvatures, not excluding zero curvature from consideration). A chain is called *oriented* if one can choose the signs of the radii consistent with the sign convention. That is, a chain is oriented if and only if the number of interior tangencies is even, see Figure 6.

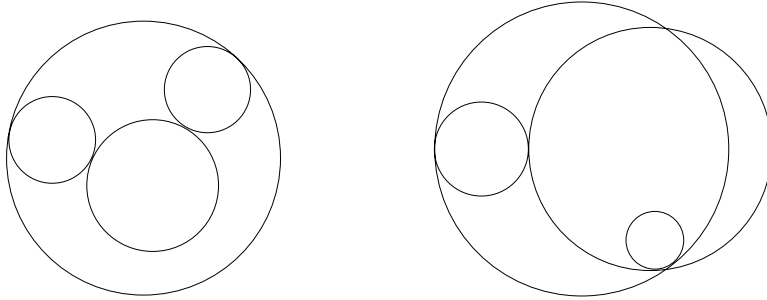


Figure 6: An oriented and a non-oriented chain of four circles

In what follows, we use half-integers as the indices for the centers of the spheres and of their radii. Consider an oriented chain of spheres with centers P_j and signed radii r_j . Denote by Q_i the tangency point of the adjacent spheres with centers $P_{i-\frac{1}{2}}$ and $P_{i+\frac{1}{2}}$. Let V_i and W_i be the two points on the line $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}$ located at distance ℓ from Q_i . The choice of labels is consistent for all i : if the segments V_iW_i and $Q_iP_{i+\frac{1}{2}}$ have the same orientations then the segments $V_{i+1}W_{i+1}$ and $Q_{i+1}P_{i+\frac{1}{2}}$ have the opposite orientations, and vice versa.

Lemma 3.1 *The polygons V and W are in the discrete bicycle correspondence. Conversely, given polygons V and W in the discrete bicycle correspondence, let $P_{i+\frac{1}{2}}$ be the intersection point of the lines $V_{i+1}W_{i+1}$ and V_iW_i . Then there exists an oriented chain of spheres centered at points P_j , such that the tangency points Q_i are the midpoints of the segments V_iW_i .*

The construction is illustrated in Figure 7.

Proof. By construction, a homothety centered at $P_{i+\frac{1}{2}}$ takes V_i to W_i and W_{i+1} to V_{i+1} . For example, in Figure 7, the homothety with the coefficient

$$-\frac{\ell - r_{\frac{3}{2}}}{\ell + r_{\frac{3}{2}}},$$

centered at $P_{\frac{3}{2}}$, takes V_1W_2 to W_1V_2 . Since $|V_iW_i| = |V_{i+1}W_{i+1}| = 2\ell$, the quadrilateral $V_iW_iV_{i+1}W_{i+1}$ is a Darboux butterfly.

Conversely, by construction,

$$|P_{i+\frac{1}{2}}W_i| = |P_{i+\frac{1}{2}}V_{i+1}|, \quad |P_{i+\frac{1}{2}}V_i| = |P_{i+\frac{1}{2}}W_{i+1}|,$$

hence $|P_{i+\frac{1}{2}}Q_i| = |P_{i+\frac{1}{2}}Q_{i+1}| := r_{i+\frac{1}{2}}$ where Q_i is the midpoint of the segment V_iW_i . The sphere with this radius passes through points Q_i and Q_{i+1} and is orthogonal to the lines $P_{i+\frac{1}{2}}Q_i$ and $P_{i+\frac{1}{2}}Q_{i+1}$. Thus one obtains a chain of spheres, and this chain is oriented. \square

The polygon Q is the discrete rear bicycle track. We apply Lemma 3.1 to prove Theorem 5.

Proof of Theorem 5. Given an oriented chain with centers at points P_j and signed radii r_j (where j is half-integer), the tangency points Q_i have the following coordinates:

$$Q_i = \frac{r_{i+\frac{1}{2}}P_{i-\frac{1}{2}} + r_{i-\frac{1}{2}}P_{i+\frac{1}{2}}}{r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}}$$

(note that this formula does not change if all radii are negated). Then the points V_i and W_i are given by the formula

$$\frac{(r_{i+\frac{1}{2}} - \ell)P_{i-\frac{1}{2}} + (r_{i-\frac{1}{2}} + \ell)P_{i+\frac{1}{2}}}{r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}}, \tag{7}$$

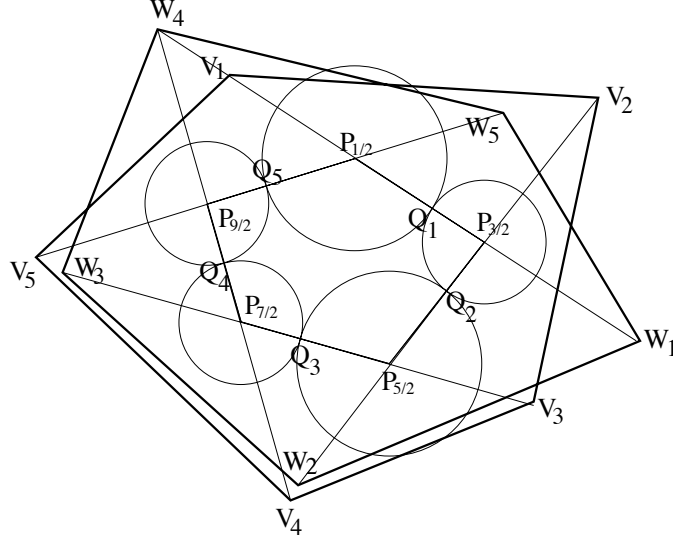


Figure 7: Polygons V and W are in the discrete bicycle correspondence

where the positive ℓ gives V_i and the negative ℓ gives W_i .

To prove the invariance of A , we need to show that A is an even function of ℓ . Indeed, using formula (7), we find that the odd (linear in ℓ) part of A is

$$\begin{aligned} & \sum \frac{(P_{i+\frac{1}{2}} - P_{i-\frac{1}{2}}) \times (r_{i+\frac{3}{2}} P_{i+\frac{1}{2}} + r_{i+\frac{1}{2}} P_{i+\frac{3}{2}})}{(r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}})(r_{i+\frac{1}{2}} + r_{i+\frac{3}{2}})} + \\ & \frac{(r_{i+\frac{1}{2}} P_{i-\frac{1}{2}} + r_{i-\frac{1}{2}} P_{i+\frac{1}{2}}) \times (P_{i+\frac{3}{2}} - P_{i+\frac{1}{2}})}{(r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}})(r_{i+\frac{1}{2}} + r_{i+\frac{3}{2}})} = \\ & \sum \frac{P_{i+\frac{1}{2}} \times P_{i+\frac{3}{2}}}{r_{i+\frac{1}{2}} + r_{i+\frac{3}{2}}} - \sum \frac{P_{i-\frac{1}{2}} \times P_{i+\frac{1}{2}}}{r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}} = 0, \end{aligned}$$

as needed.

To prove that J is invariant, one makes a similar computation. Let e_i be the unit vector from Q_i to $P_{i+\frac{1}{2}}$.

One has $V_i = Q_i + \ell e_i$ and $W_i = Q_i - \ell e_i$. Hence

$$|V_i|^2 = \ell^2 + 2\ell Q_i \cdot e_i + |Q_i|^2.$$

It follows that

$$|V_{i+1}|^2 - |V_{i-1}|^2 = |Q_{i+1}|^2 - |Q_{i-1}|^2 + 2\ell (Q_{i+1} \cdot e_{i+1} - Q_{i-1} \cdot e_{i-1}),$$

and the odd (linear in ℓ) part of J is

$$\sum (|Q_{i+1}|^2 - |Q_{i-1}|^2) e_i + 2(Q_{i+1} \cdot e_{i+1} - Q_{i-1} \cdot e_{i-1}) Q_i. \quad (8)$$

Rewrite negative (8) as

$$\begin{aligned} & \sum |Q_i|^2 (e_{i+1} - e_{i-1}) + 2Q_i \cdot e_i (Q_{i+1} - Q_{i-1}) = \\ & \sum |Q_i|^2 ((e_{i+1} + e_i) - (e_i + e_{i-1})) + 2Q_i \cdot e_i (Q_{i+1} - Q_{i-1}). \end{aligned} \quad (9)$$

Using the formulas

$$\begin{aligned} Q_{i-1} &= Q_i - r_{i-\frac{1}{2}}(e_i + e_{i-1}), \quad Q_{i+1} = Q_i + r_{i+\frac{1}{2}}(e_i + e_{i+1}), \\ P_{i-\frac{1}{2}} &= Q_i - r_{i-\frac{1}{2}}e_i, \quad P_{i+\frac{1}{2}} = Q_i + r_{i+\frac{1}{2}}e_i, \end{aligned}$$

rewrite (9) as

$$\begin{aligned} & \sum (|Q_i|^2 + 2r_{i+\frac{1}{2}}Q_i \cdot e_i)(e_{i+1} + e_i) - (|Q_i|^2 - 2r_{i-\frac{1}{2}}Q_i \cdot e_i)(e_{i-1} + e_i) = \\ & \sum (|P_{i+\frac{1}{2}}|^2 - r_{i+\frac{1}{2}}^2)(e_{i+1} + e_i) - \sum (|P_{i-\frac{1}{2}}|^2 - r_{i-\frac{1}{2}}^2)(e_{i-1} + e_i) = 0, \end{aligned}$$

as needed. \square

Remark 3.2 One has the following relation between the integrals A and J :

$$D_\xi(J)(V) = -2A(V) \cdot \xi = 2 \sum_i (V_i \cdot \xi) (V_{i-1} - V_{i+1}), \quad (10)$$

where D_ξ is the directional derivative along a vector ξ and where dot is the Euclidean pairing of 2-vectors and vectors. Of course, (10) is also an integral for every vector ξ .

Remark 3.3 The integral A is invariant under parallel translations, but J is neither invariant under parallel translations nor commutes with them. In dimension two, we adjust the integral J so that it commutes with parallel translations and thus becomes a “center”, associated with a polygon. Namely, rotate $J(V)$ through 90° and divide by four times the area:

$$\frac{1}{4A(V)} \left(\sum (y_i^2 y_{i+1} - y_i y_{i+1}^2 + x_i^2 y_{i+1} - x_{i+1}^2 y_i), \sum (x_i x_{i+1}^2 - x_i^2 x_{i+1} + x_i y_{i+1}^2 - x_{i+1} y_i^2) \right),$$

where $V_i = (x_i, y_i)$ and the sums are cyclic. We call this point the *circumcenter of mass* of the polygon V and denote it by $CCM(V)$.

A justification of this terminology is as follows. Consider a triangulation of the polygon V , and let O_i be the circumcenter of i th triangle. Then $CCM(V)$ is the center of mass of the points O_i , taken with the weight equal to the (oriented) area of i th triangle. The result does not depend on triangulation. This construction is mentioned in [1]; we plan to study it in detail in a forthcoming paper [17].

Let us mention, without proof, two properties of $CCM(V)$. First, if V is an equilateral polygon then the circumcenter of mass coincides with the center of mass. This agrees with the observation, made in [4] that, in our terminology, the discrete bicycle transformation of an equilateral polygon preserves its center of mass.

Second, in the continuous limit, as V becomes a curve γ , the circumcenter of mass of V tends to the center of mass of the homogeneous lamina bounded by γ . As a consequence, the continuous bicycle transformation preserves the center of mass.

We plan to study the monodromy integrals in a separate paper. We comment on these integrals in dimension two in the next section.

4 In the plane

In this section, we consider the discrete bicycle transformation in the plane. We start with a simple observation: for an inscribed polygon, a rotation about the circumcenter is a discrete bicycle transformation, see Figure 8.

Our first result concerns convex inscribed polygons.

Theorem 6 *Let V be a convex inscribed polygon, and let d be the diameter of the circumcircle. The discrete bicycle monodromy $M_{V,\ell}$ is elliptic for $\ell > d$, parabolic for $\ell = d$, and hyperbolic for $\ell \in (0, d)$. In the last case, the discrete bicycle transformation is a rotation about the circumcenter.*

Proof. As we mentioned, if $\ell \in (0, d)$ then a rotation about the circumcenter is a discrete bicycle transformation.

Let $W = T_\ell(V)$. Then W has the same perimeter and the same oriented area as V , see Theorem 5. It is known that, among polygons with given side

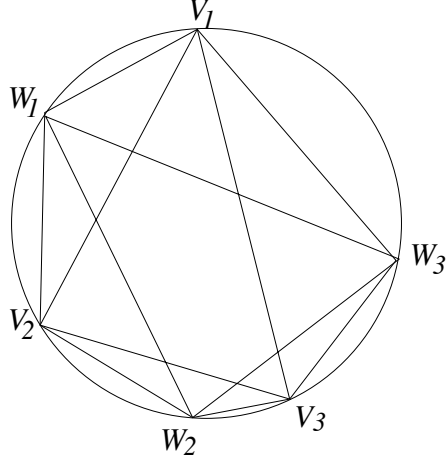


Figure 8: Triangles V and W are in the discrete bicycle correspondence

lengths, there exists a unique area maximizing one, and this is an inscribed convex polygon. It follows that W is inscribed, and hence congruent to W . It follows from Theorem 5 that the circumcenter of W coincides with that of V , see Remark 3.3. It follows that W is a rotation of V about the circumcenter. \square

In particular, Theorem 6 completely described the discrete bicycle transformation on triangles.

Next we consider a $2k$ -gon whose sides lie, in an alternating fashion, on two concentric circles. In the limiting case, the two concentric circles may become two parallel lines.

Proposition 4.1 *Let C_1 and C_2 be concentric circles with the center O (or parallel lines). Let the odd vertices of a $2k$ -gon lie on C_1 and the even ones on C_2 . Let W_1 be a point of C_2 . Then the discrete bicycle transformation of V with the initial segment V_1W_1 is a closed $2k$ -gon whose odd vertices lie on C_2 and the even ones on C_1 . The second iteration of this discrete bicycle transformation sends V to an isometric polygon.*

Proof. Reflect V_1 in the perpendicular bisector of the segment W_1V_2 to obtain W_2 , and continue in the same way, see Figure 9. Let the lower case letters denote the angular coordinates of the respective points. Then

$$w_2 = w_1 + v_2 - v_1, \quad w_3 = w_2 + v_3 - v_2 = w_1 + v_3 - v_1,$$

etc. It follows that $w_{2k+1} = w_1 + v_{2k+1} - v_1 = w_1$, hence the polygon W is closed.

We see that the discrete bicycle transformation \mathcal{T} is the composition of two commuting transformations: the rotation through the angle $w_1 - v_1$, and the involution that interchanges the points of C_1 and C_2 on the same radial ray. Hence \mathcal{T}^2 is a rotation.

The argument for parallel lines is analogous, and the resulting polygon W is obtained from V by a glide reflection. In this case, the orbit of the polygon is unbounded. \square

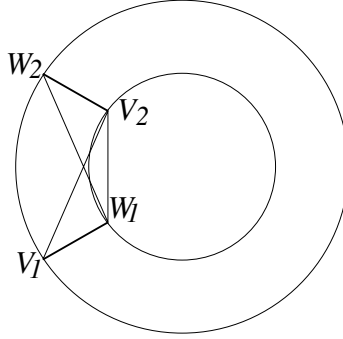


Figure 9: W_2 is the reflection of V_1 in the perpendicular bisector of W_1V_2

Note that, in this construction, the polygon Q , whose vertices are the midpoints of the segments V_iW_i (see Lemma 3.1), is inscribed in a circle with the center O . We also have the following consequence of the proof.

Corollary 4.2 *If the polygon in the preceding Proposition is a rhombus then its image under the bicycle transformation is a congruent rhombus.*

Now we discuss the monodromy integrals for plane polygons. The monodromy along a side is given by formula (3); the full monodromy M is the product of these monodromies over the consecutive sides of the polygon. The monodromy is defined only up to a multiplicative factor, and the invariant quantity is

$$\frac{\text{Tr}^2(M)}{\det(M)},$$

considered as a function of ℓ . Note that the determinant of the matrix (3) equals $\ell^2 - a^2$, that is, is also an integral. Thus $\text{Tr}(M)$ is an integral.

Proposition 4.3 *Consider a k -gon whose sides have the lengths a_1, \dots, a_k and the directions $\alpha_1, \dots, \alpha_k$. Then*

$$\text{Tr}(M) = 2(\ell^k + c_1\ell^{k-1} + c_2\ell^{k-2} + \dots + c_k)$$

with all odd coefficients c_1, c_3, \dots equal to zero. If k is even then the free term c_k equals

$$a_1 \dots a_k \cos(\alpha_1 - \alpha_2 + \dots - \alpha_k).$$

One also has:

$$c_2 = -\frac{1}{2} \sum a_i^2.$$

Proof. One has

$$M = \prod_{i=1}^k (\ell E + a_i A(\alpha_i))$$

where

$$A(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & -\cos \alpha \end{pmatrix}.$$

Therefore

$$\text{Tr}(M) = \sum_{j=0}^k \ell^{k-j} a_{i_1} \dots a_{i_j} \text{Tr}(A(\alpha_{i_1}) \dots A(\alpha_{i_j})).$$

Notice that

$$A(\alpha)A(\beta) = \begin{pmatrix} \cos(\alpha - \beta) & \sin(\alpha - \beta) \\ -\sin(\alpha - \beta) & \cos(\alpha - \beta) \end{pmatrix}, \quad (11)$$

a rotation matrix. More generally, the product of an odd number of the matrices $A(\alpha_i)$ is traceless, and the product of an even number is a rotation through the alternating sum of the respective angles. This implies the first two claims.

For the last claim, let u_1, \dots, u_k be the vectors of the sides of the polygon. Using (11), we find that

$$c_2 = \sum_{i < j} u_i \cdot u_j.$$

One has: $\sum u_i = 0$. Taking dot with itself yields:

$$0 = \sum u_i \cdot u_i + 2 \sum_{i < j} u_i \cdot u_j.$$

Thus

$$c_2 = -\frac{1}{2} \sum a_i^2,$$

as claimed. \square

Corollary 4.4 *The quantity $\cos(\alpha_1 - \alpha_2 + \cdots - \alpha_k)$ is an integral of the discrete bicycle transformation on even-gons.*

Let polygons V and W be in the discrete bicycle correspondence. Let

$$\alpha_i = \angle V_{i-1}V_iW_i = \angle V_{i-1}W_{i-1}W_i,$$

see Figure 10. If one knows the cyclic sequence of angles α_i then one can construct W from V : indeed, the lengths of all the segments V_iW_i are equal to 2ℓ .

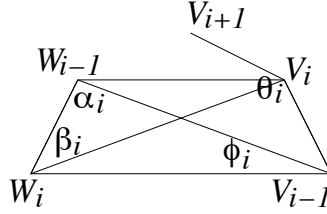


Figure 10: Notations for Proposition 4.5

The angles α_i satisfy a first order nonlinear difference equation with periodic coefficients. Let $\theta_i = \angle V_{i-1}V_iV_{i+1}$ and $c_i = |V_{i-1}V_i|$.

Proposition 4.5 *One has*

$$2\ell \cos\left(\frac{\alpha_i - \alpha_{i-1} + \theta_{i-1}}{2}\right) = c_i \cos\left(\frac{\alpha_i + \alpha_{i-1} - \theta_{i-1}}{2}\right). \quad (12)$$

Proof. Let

$$\beta_i = \angle W_{i-1}V_{i-1}V_i = \angle W_{i-1}W_iV_i, \quad \phi_i = \angle W_{i-1}V_{i-1}W_i = \angle V_iW_iV_{i-1}.$$

Then $2\phi_i = \pi - \alpha_i - \beta_i$. Since $\angle W_iV_iV_{i+1} = \beta_{i+1}$, one has $\beta_{i+1} = \theta_i - \alpha_i$. Therefore

$$\phi_i = \frac{\pi}{2} - \frac{\alpha_i - \alpha_{i-1} + \theta_{i-1}}{2}, \quad \beta_i + \phi_i = \frac{\pi}{2} - \frac{\alpha_{i-1} + \alpha_i - \theta_{i-1}}{2}. \quad (13)$$

By Sine Rule in triangle $V_{i-1}V_iW_i$,

$$\frac{2\ell}{\sin(\beta_i + \phi_i)} = \frac{c_i}{\sin \phi_i},$$

or

$$2\ell \cos\left(\frac{\alpha_i - \alpha_{i-1} + \theta_{i-1}}{2}\right) = c_i \cos\left(\frac{\alpha_i + \alpha_{i-1} - \theta_{i-1}}{2}\right),$$

as claimed. \square

As an application of Proposition 4.5, we compute the eigenvalue of the fixed point of the monodromy map of the polygon V corresponding to the pair of polygons V, W in the discrete bicycle correspondence. Since the monodromy is a Möbius transformation, the eigenvalues of its two fixed points are reciprocals of each other.

Theorem 7 *The eigenvalue in question equals*

$$\prod_{i=1}^n \frac{|V_{i-1}W_i|}{|V_iW_{i-1}|} = \prod_{j=1/2}^{n+1/2} \frac{|\ell + r_j|}{|\ell - r_j|}.$$

In particular, the monodromy is parabolic if and only if

$$\prod_{i=1}^n |V_{i-1}W_i| = \prod_{i=1}^n |V_iW_{i-1}| \quad \text{or} \quad \prod_{j=1/2}^{n+1/2} |\ell + r_j| = \prod_{j=1/2}^{n+1/2} |\ell - r_j|.$$

Proof. To compute the eigenvalue, one linearizes equation (12): if u_i is a variation of α_i then the linearization is as follows:

$$2\ell(u_i - u_{i-1}) \sin\left(\frac{\alpha_i - \alpha_{i-1} + \theta_{i-1}}{2}\right) = c_i(u_i + u_{i-1}) \sin\left(\frac{\alpha_i + \alpha_{i-1} - \theta_{i-1}}{2}\right),$$

and hence

$$\begin{aligned} & u_i \left[2\ell \sin\left(\frac{\alpha_i - \alpha_{i-1} + \theta_{i-1}}{2}\right) - c_i \sin\left(\frac{\alpha_i + \alpha_{i-1} - \theta_{i-1}}{2}\right) \right] = \\ & u_{i-1} \left[2\ell \sin\left(\frac{\alpha_i - \alpha_{i-1} + \theta_{i-1}}{2}\right) + c_i \sin\left(\frac{\alpha_i + \alpha_{i-1} - \theta_{i-1}}{2}\right) \right]. \end{aligned}$$

By elementary geometry of the trapezoid in Figure 4.5 and formulas (13), one has:

$$\begin{aligned} 2\ell \sin\left(\frac{\alpha_i - \alpha_{i-1} + \theta_{i-1}}{2}\right) &= \frac{1}{2}(|V_{i-1}W_i| + |V_iW_{i-1}|), \\ c_i \sin\left(\frac{\alpha_i + \alpha_{i-1} - \theta_{i-1}}{2}\right) &= \frac{1}{2}(|V_{i-1}W_i| - |V_iW_{i-1}|). \end{aligned}$$

Therefore

$$u_i |V_i W_{i-1}| = u_{i-1} |V_{i-1} W_i|,$$

which implies the first formula for the eigenvalue.

For the second formula, note that a homothety centered at point $P_{i+1/2}$ takes segment $V_i W_{i+1}$ to segment $V_{i+1} W_i$, see Figure 7. The coefficient of this homothety is $|\ell + r_{i+1/2}|/|\ell - r_{i+1/2}|$, and we obtain the second formula for the eigenvalue.

It remains to notice that the monodromy is parabolic if and only if the two reciprocal eigenvalues coincide. \square

Remark 4.6 The continuous analogs of Proposition 4.5 and Theorem 7 are contained in [13]. Namely, the continuous version of (12) is the differential equation

$$\frac{d\alpha}{dx} + \frac{\sin \alpha}{\ell} = \kappa(x)$$

where $\alpha(x)$ is the angle made by the bicycle frame with the front wheel trajectory, x is the arc length parameter along this trajectory, and $\kappa(x)$ is the curvature of this curve. The endpoint of the segment of length ℓ describes the rear wheel trajectory.

The continuous version of Theorem 7 states that the eigenvalues of the bicycle monodromy are $e^{\pm \text{length}(\gamma)}$ where γ is the rear wheel trajectory, and the length is algebraic: the sign changes after one traverses a cusp. In particular, the monodromy is parabolic if and only if the rear track has zero length.

5 Case study: plane quadrilaterals

In this section, we describe the dynamics of the discrete bicycle transformation on plane quadrilaterals.

We have a trichotomy according to the position of the circumcenter of mass, see Remark 3.3. Consider a quadrilateral $ABCD$. The first case is when the diagonals AC and BD are not parallel. Let O be the intersection point of the perpendicular bisectors of these diagonals, see Figure 11 on the left.

Lemma 5.1 O is the circumcenter of mass of the quadrilateral $ABCD$.

Proof. The circumcenters of the triangles ABD and BCD lie on the perpendicular bisector of the segment BD , and the circumcenters of the triangles ABC and ACD lie on the perpendicular bisector of the segment AC . Hence $O = CCM(ABCD)$. \square

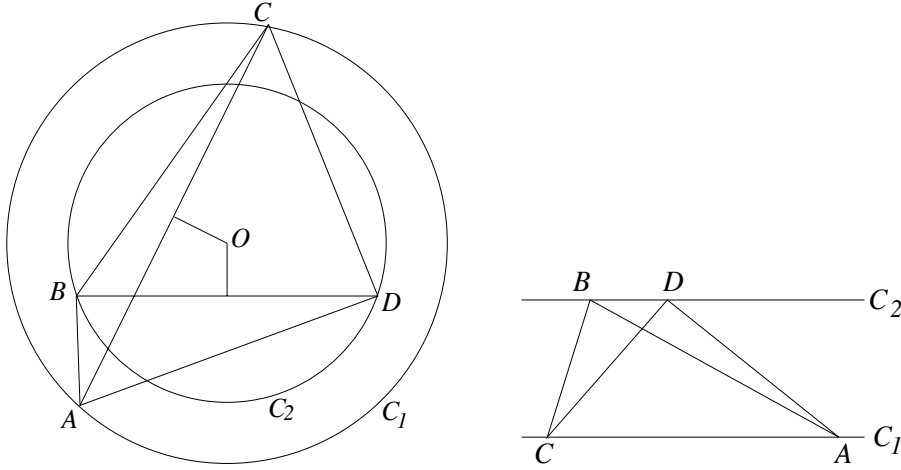


Figure 11: Two types of quadrilaterals: the circumcenter is finite or infinite

In the first case, A and C lie on one circle, say, C_1 , and B and D on another circle, C_2 , centered at O . Denote their radii by r_1 and r_2 , and assume that $r_1 \geq r_2$.

The second case is when the diagonals are parallel but the quadrilateral is not a Darboux butterfly, see Figure 11 on the right. In this case, the two concentric circles are replaced by two parallel lines, and the center O is at infinity. Although both radii are infinite, their difference $r_1 - r_2$ is still defined and equals the distance between the parallel lines. Note that, in this case, the quadrilateral $ABCD$ has zero area.

The third case is when the quadrilateral is a Darboux butterfly. In this case, there exists an infinite family of pairs of concentric circles C_1, C_2 such that $A, C \in C_1$ and $B, D \in C_2$. The centers of these circles lie on the common perpendicular bisector of the segments AC and BD , including the point at infinity, when the circles become parallel lines.

Theorem 8 *Let $ABCD$ be a quadrilateral. If $ABCD$ is not a Darboux butterfly then the discrete bicycle monodromy about the quadrilateral is elliptic for $\ell \in (0, r_1 - r_2) \cup (r_1 + r_2, \infty)$, hyperbolic for $\ell \in (r_1 - r_2, r_1 + r_2)$, and parabolic for $\ell = r_1 \pm r_2$. For ℓ in the hyperbolic or parabolic range, the discrete bicycle correspondence is induced by a point $A' \in C_2$, as described in Proposition 4.1. If $ABCD$ is a Darboux butterfly then the monodromy is the identity. For every starting point A' , there exists a circle (or straight line) C_2 that passes through A' , and the discrete bicycle correspondence is again described by Proposition 4.1.*

Proof. If $\ell \in [r_1 - r_2, r_1 + r_2]$ then there exist two points $A' \in C_2$ such that $|AA'| = \ell$ (these two points coincide for $\ell = r_1 \pm r_2$), and Proposition 4.1 describes the discrete bicycle transformation.

Conversely, assume that $A'B'C'D'$ is a discrete bicycle transformation of $ABCD$. Let l_1, l_2, l_3 and l_4 be the perpendicular bisectors of the segments $A'B, B'C, C'D$ and $D'A$, respectively. Let R_i be the reflection in the line l_i , $i = 1, 2, 3, 4$. By definition of the bicycle monodromy,

$$B' = R_1(A), \quad C' = R_2(B), \quad D' = R_3(C), \quad A' = R_4(D),$$

see Figure 9. Note also that

$$B = R_1(A'), \quad C = R_2(B'), \quad D = R_3(C'), \quad A = R_4(D').$$

We claim that the lines l_1, l_2, l_3, l_4 are concurrent (as a particular case, the four lines may be parallel).

Consider the composition $F = R_3 \circ R_2 \circ R_1$: it is either a reflection or a glide reflection. We claim that the former is the case. Two given congruent line segments $AA', D'D$ are related by just one odd isometry. Since $AA'D'D$ is an isosceles trapezoid, this isometry is a reflection.

Since $R_3 \circ R_2 \circ R_1$ is a reflection, the lines l_1, l_2 and l_3 are concurrent. Applying the same argument to l_2, l_3, l_4 , we conclude that all four lines are concurrent.

To fix ideas, let us assume that the intersection point of the lines l_1, l_2, l_3, l_4 is finite (the case of parallel lines is similar). Denote this point by Q . We claim that $Q = O$, the circumcenter of the quadrilateral $ABCD$.

Indeed, $R_2 \circ R_1(A) = C$, hence Q lies on the perpendicular bisector of the diagonal AC . Likewise, $R_3 \circ R_2(B) = D$, hence Q lies on the perpendicular bisector of the diagonal BD . Thus $Q = O$.

Since $A' = R_1(B)$, it follows that $A' \in C_2$, and we are in the situation of Proposition 4.1.

It remains to consider the case of a Darboux butterfly. For any starting point A' , we can find a circle C_2 through A', B and D with the center O on the perpendicular bisector of the segments AC and BD . Then another circle C_1 , centered at O , passes through A and C , and we are in the situation described in Proposition 4.1. \square

Remark 5.2 The preceding argument provides an alternative proof of the fact that the monodromy of a Darboux butterfly is the identity for all ℓ .

We now discuss an application of Theorem 8 to the following problem in “bicycle mathematics”. Suppose one is given two closed curves, the front and rear bicycle tracks. Can one always determine in which direction the bicycle went? Usually, one can, but sometimes one cannot: consider, for example, two concentric circles.

Describing such pairs of “ambiguous” bicycle tracks is an interesting and difficult problem, and only partial results are available. This problem is equivalent to Ulam’s problem of describing uniform (2-dimensional) bodies that float in equilibrium in all positions. We refer to [3, 4, 8, 18, 19] for the literature on this intriguing topic.

A discrete version of this problem was introduced in [15]. Define a *bicycle* (n, k) -gon as an equilateral n -gons whose k -diagonals have equal length. More precisely, if the polygon is $V_1 V_2 \dots V_n$ then we require that $V_i V_{i+1} V_{i+k+1} V_{i+k}$ be a Darboux butterfly for all i (as usual, the indices are understood cyclically). The problem is to describe bicycle (n, k) -gons, in particular, to determine for which pairs (n, k) such a polygon must be regular. See also [6, 7].

For example, it is shown in [15] that bicycle $(n, 2)$ -gons, $(2k + 1, k)$ -gons, and $(3k, k)$ -gons are regular. On the other hand, an example of a non-regular bicycle polygon is shown in Figure 12. This construction generalizes to all pairs (n, k) and yields 1-parameter families of bicycle (n, k) -gons with even n

and odd k . Note that the even and the odd vertices of a polygon in Figure 12 lie on two concentric circles and that the polygons have dihedral symmetry.

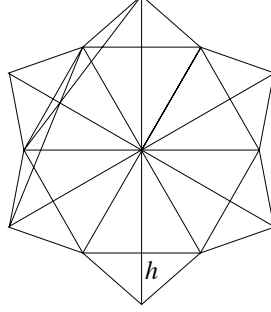


Figure 12: A bicycle $(12, 3)$ -gon: h is a parameter of the construction

Let ℓ be the length of the k -diagonal of a bicycle (n, k) -gon, and let S be the cyclic relabeling of the vertices: $V_i \mapsto V_{i+1}$. One can restate the definition in terms of the discrete bicycle transformation \mathcal{T}_ℓ : V is a bicycle (n, k) -gon if $\mathcal{T}_\ell(V) = S^k(V)$.

The next result is a further step toward classification of bicycle polygons.

Theorem 9 *If k is even then a bicycle $(4k, k)$ -gon is regular. If k is odd then the even vertices of a bicycle $(4k, k)$ -gon are equally spaced on a circle and its odd vertices are equally spaced on a concentric circle, that is, the polygon is obtained from a regular $2k$ -gon by the construction depicted in Figure 12.*

Proof. Given a bicycle $(4k, k)$ -gon V , consider the rhombus $V_0V_kV_{2k}V_{3k}$. The discrete bicycle transformation with the length parameter V_0V_1 takes this rhombus to $V_1V_{k+1}V_{2k+1}V_{3k+1}$, to $V_2V_{k+2}V_{2k+2}V_{3k+2}$, and so on.

Let O be the center of the rhombus $V_0V_kV_{2k}V_{3k}$, and let C_1 and C_2 be the concentric circles centered at O such that $V_0, V_{2k} \in C_1$ and $V_k, V_{3k} \in C_2$. By Corollary 4.2, all the consecutive rhombi are congruent, and $V_1 \in C_2, V_2 \in C_1, V_3 \in C_2, V_4 \in C_1$, etc.

Therefore, if k is even, then $V_k \in C_1$, and hence $C_1 = C_2$. It follows that the rhombus is a square and V is a regular $4k$ -gon. If k is odd then the even vertices of V form a regular $2k$ -gon inscribed into C_1 , and the odd ones form a regular $2k$ -gon inscribed into C_2 . Thus V is obtained from a regular $2k$ -gon by the construction in Figure 12. \square

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